International Journal of Theoretical Physics, Vol. 2, No. 4 (1969), pp. 381–403

On the Geometric Foundations of Nuclear Shell Structure

R. RAMANNA AND S. JYOTHI

Bhabha Atomic Research Centre,
Trombay, Bombay, India

Received: 13 *May* 1969

Abstract

Cartan's geometric theory of partial differential equations is applied to a system of Schrödinger equations. It is shown that the choice of a Riemann manifold which is a torus is equivalent to using a many-body neutron and proton potential commonly used in nuclear theory. The theory is applied to spinless, ground-state systems using the Dirichlet principle to minimise the energy, to obtain the neutron–proton ratios, Coulomb and binding energies of nuclei. A shell structure naturally manifests itself from the choice of the manifold.

In earlier papers (Ramanna, 1968a, b) the geometric theory of partial differential equations due to Cartan (Hermann, 1965) was applied to a system of Schrödinger equations to determine its characteristics by the method of determining an appropriate manifold of the exterior differential system associated with the partial differential equations. It has been shown (Ramanna, 1968a, b) that several nuclear characteristics of spinless, ground-state systems, such as Coulomb energies, binding energies and shell structure, can be obtained by assuming the manifold to be a torus which is also a Riemann surface. The Dirichlet principle was used to minimise the energy. In this investigation we summarise the previous results and show that this manifold corresponds exactly to the many-body neutron and proton nuclear potential commonly assumed in nuclear theory. The chosen geometry of the partial differential equations predicts a strong repulsion at small distances for both nucleons and a Coulomb term for protons. The potentials obtained in this manner give the potentials in a many-body system, as the entire assembly is taken into account by the nature of the solutions.

Consider two bounded quantum mechanical systems described by the following two Schrödinger equations each defined over a domain,

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi_1(x_1,t_1)}{\partial x_1{}^2} + V_1(x_1,t_1)\,\psi_1(x_1,t_1) = i\hbar\frac{\partial\psi_1(x_1,t_1)}{\partial t_1} \tag{1}$$

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi_2(x_2,t_2)}{\partial x_2{}^2} + V_2(x_2,t_2)\,\psi_2(x_2,t_2) = i\hbar\frac{\partial\psi_2(x_2,t_2)}{\partial t_2} \tag{2}$$

Here V_1 and V_2 are taken to be negative.

The two systems have the same mass and are considered to be spherically symmetrical, so only one coordinate is used to describe the space. The coordinate system of each domain is locally Euclidean, and in general the two equations represent two different problems. If, however, the domains overlap in a specified way, the two equations hold simultaneously in the region of overlap and this invariance implies that ψ_1 and ψ_2 are related to one another. If the region of overlap is a plane, the ψ's become identical and the two equations represent the same system. If, however, the overlap is over a surface which is not a plane, a more complicated relation exists between the coordinate systems which couple the two equations. A surface over which locally Euclidean coordinate systems overlap is called a manifold (Flanders, 1963). We show that the form of the relations connecting the coordinate systems which define the manifold also determines the nature of the nuclear potential inside a many-body system. The manifold used by Ramanna (1968a, b) gives rise, in special cases, to a neutron and proton potential having a strong repulsive term at short distances and a Coulomb term in the case of the latter, thereby substantiating the successes of the earlier work.

We now restrict ourselves to harmonic transformations, i.e. to such transformations which obey the Cauchy-Riemann relations because of their unique property that the total energy of the system is minimised. It is shown later, and by Ramanna (1968a, b), that for systems obeying Schrödinger equations and for transformations defined below, the total energy is automatically minimised through the Dirichlet principle.

Let the transformation law relating to the coordinate systems be given by the Cauchy-Riemann relations

$$\frac{\partial x_1}{\partial x_2}=\frac{\partial t_1}{\partial t_2} \quad \text{and} \quad \frac{\partial x_1}{\partial t_2}=-\frac{\partial t_1}{\partial x_2} \tag{3}$$

where x and t are along the real and imaginary axes, respectively. A manifold over which such coordinate systems are defined is called a two-dimensional Riemann surface.

We define

$$p=\frac{\partial x_1}{\partial x_2}=\frac{\partial t_1}{\partial t_2}$$

$$q=-\frac{\partial x_1}{\partial t_2}=\frac{\partial t_1}{\partial x_2} \tag{3a}$$

If one assumes that (x_1, t_1) and (x_2, t_2) are independent of each other, i.e.

$$\frac{\partial x_1}{\partial t_1}=0, \quad \frac{\partial x_2}{\partial t_2}=0, \quad \frac{\partial t_1}{\partial x_1}=0 \quad \text{and} \quad \frac{\partial t_2}{\partial x_2}=0$$

then by straightforward algebra, and using the chain rule for derivatives, it can be shown that the following relations also hold:

$$\frac{\partial p}{\partial x_1} = \frac{\partial q}{\partial t_1}, \qquad \frac{\partial p}{\partial t_1} = -\frac{\partial q}{\partial x_1} \tag{4}$$

Let $\psi_2(x_2, t_2) = \omega(x_1, t_1)\psi_1(x_1, t_1)$ where $\omega(x_1, t_1)$ is a complex valued function of x_1 and t_1. This implies that the wave function in one domain looks distorted in space and time when observed from the coordinate system of the other domain.

Using the coordinate transformation of the (x_2, t_2) into (x_1, t_1) as prescribed in (3) and (4), and transforming equation (2) to the (x_1, t_1) coordinate system, we get

$$\begin{aligned}
&\frac{\partial^2\psi_1}{\partial x_1^2} + \frac{\partial\psi_1}{\partial x_1}\left[\frac{2}{\omega}\frac{\partial\omega}{\partial x_1} + \frac{2q}{p}\frac{1}{\omega}\frac{\partial\omega}{\partial t_1} + \frac{1}{p}\frac{\partial p}{\partial x_1} + \frac{q}{p^2}\frac{\partial p}{\partial t_1} + \frac{2mqi}{\hbar p^2}\right]\\
&\quad + \frac{2q}{p}\frac{\partial^2\psi_1}{\partial x_1\,\partial t_1} + \frac{q^2}{p^2}\frac{\partial^2\psi_1}{\partial t_1^2} + \frac{\partial\psi_1}{\partial t_1}\left[2\frac{q}{p}\frac{1}{\omega}\frac{\partial\omega}{\partial x_1} + \frac{2q^2}{p^2}\frac{1}{\omega}\frac{\partial\omega}{\partial t_1}\right.\\
&\quad \left. -\frac{1}{p}\frac{\partial p}{\partial t_1} + \frac{q}{p^2}\frac{\partial p}{\partial x_1} - \frac{2mi}{\hbar p}\right] + \psi_1\left[\frac{1}{\omega}\frac{\partial^2\omega}{\partial x_1^2} + \frac{2q}{p}\frac{1}{\omega}\frac{\partial^2\omega}{\partial x_1\,\partial t_1}\right.\\
&\quad + \frac{q^2}{p^2}\frac{1}{\omega}\frac{\partial^2\omega}{\partial t_1^2} + \frac{1}{\omega}\frac{\partial\omega}{\partial x_1}\left(\frac{1}{p}\frac{\partial p}{\partial x_1} + \frac{q}{p^2}\frac{\partial p}{\partial t_1} + \frac{2mqi}{\hbar p^2}\right)\\
&\qquad\qquad \left. + \frac{1}{\omega}\frac{\partial\omega}{\partial t_1}\left(-\frac{1}{p}\frac{\partial p}{\partial t_1} + \frac{q}{p^2}\frac{\partial p}{\partial x_1} - \frac{2mi}{\hbar p}\right) + \frac{2mV_2}{\hbar^2 p^2}\right] = 0
\end{aligned} \tag{5}$$

Comparing this equation with the Schrödinger equation (1) in the region of overlap, the following relations hold:

$$2\frac{q}{p}\frac{1}{\omega}\frac{\partial\omega}{\partial x_1} + 2\frac{q^2}{p^2}\frac{1}{\omega}\frac{\partial\omega}{\partial t_1} - \frac{1}{p}\frac{\partial p}{\partial t_1} + \frac{q}{p^2}\frac{\partial p}{\partial x_1} - \frac{2mi}{\hbar p} = \frac{-2mi}{\hbar} \tag{6}$$

$$\begin{aligned}
&\frac{1}{\omega}\frac{\partial^2\omega}{\partial x_1^2} + \frac{2q}{p}\frac{1}{\omega}\frac{\partial^2\omega}{\partial x_1\,\partial t_1} + \frac{q^2}{p^2}\frac{1}{\omega}\frac{\partial^2\omega}{\partial t_1^2} + \frac{1}{\omega}\frac{\partial\omega}{\partial x_1}\left(\frac{1}{p}\frac{\partial p}{\partial x_1} + \frac{q}{p^2}\frac{\partial p}{\partial t_1} + \frac{2mqi}{\hbar p^2}\right)\\
&\qquad + \frac{1}{\omega}\frac{\partial\omega}{\partial t_1}\left(-\frac{1}{p}\frac{\partial p}{\partial t_1} + \frac{q}{p^2}\frac{\partial p}{\partial x_1} - \frac{2mi}{\hbar p}\right) + \frac{2m}{\hbar^2 p^2}(V_2 - p^2 V_1) = 0
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\frac{\partial\psi_1}{\partial x_1}\left[\frac{2}{\omega}\frac{\partial\omega}{\partial x_1} + \frac{2q}{p}\frac{1}{\omega}\frac{\partial\omega}{\partial t_1} + \frac{1}{p}\frac{\partial p}{\partial x_1} + \frac{q}{p^2}\frac{\partial p}{\partial t_1} + \frac{2mqi}{\hbar p^2}\right]\\
&\qquad\qquad + \frac{2q}{p}\frac{\partial^2\psi_1}{\partial x_1\,\partial t_1} + \frac{q^2}{p^2}\frac{\partial^2\psi_1}{\partial t_1^2} = 0
\end{aligned} \tag{8}$$

In order to understand the physical implications of these transformations, we consider the following approximations corresponding to a time-independent case:

$$\left.\begin{aligned} &\text{(a)} \quad \frac{\partial \omega}{\partial t_1} = 0 \\ \text{and} \\ &\text{(b)} \quad \frac{\partial p}{\partial t_1} = 0 \end{aligned}\right\} \tag{9}$$

(a) implies that the wave functions in one system appear distorted only spatially when viewed from the other, and (b) implies that any solution of the equations must make p independent of t and yet satisfy equation (4). We see later that the doubly-periodic functions of Weierstrass satisfy this requirement.

Eliminating ω from equations (7) and (8), and separating the real and imaginary parts, we get (Appendix):

$$p\frac{\partial^2 p}{\partial x_1{}^2} - \frac{1}{2}\left(\frac{\partial p}{\partial x_1}\right)^2 + \frac{2m^2}{\hbar^2 q^2}(p-1)(p^3-p^2-2q^2) = \frac{4m}{\hbar^2}(V_2{}^r - p^2 V_1{}^r) \tag{10}$$

$$-\frac{2m}{\hbar pq}\left(p^3 - 2p^2 - q^2\right)\frac{\partial p}{\partial x_1} = \frac{4m}{\hbar^2}(V_2{}^i - p^2 V_1{}^i) \tag{11}$$

where $V_2{}^r$, $V_1{}^r$ and $V_2{}^i$, $V_1{}^i$ are the real and imaginary parts of the respective potentials. If the potentials under consideration are purely real, the imaginary part can be put equal to zero, i.e.

$$V_2{}^i - p^2 V_1{}^i = 0$$

Therefore,

$$\frac{2m}{\hbar pq}\left(p^3 - 2p^2 - q^2\right)\frac{\partial p}{\partial x_1} = 0$$

or

$$p^3 - 2p^2 = q^2 \tag{11a}$$

Eliminating q from equation (10) we get

$$p\frac{\partial^2 p}{\partial x_1{}^2} - \frac{1}{2}\left(\frac{\partial p}{\partial x_1}\right)^2 + \frac{2m^2(p-1)(3-p)}{\hbar^2(p-2)} = \frac{4m}{\hbar^2}(V_2 - p^2 V_1) \tag{12}$$

This is a standard nonlinear differential equation in p whose solutions are usually automorphic functions (Davis, 1961). The solution must also satisfy (4) and (9b), which is true in the case of automorphic functions.

We therefore assume

$$\begin{aligned} p + iq &= A\exp\{-\wp[(x_1 + it_1), g_2, g_3]\} + B + iC(t_1) \\ p - iq &= A\exp\{-\wp[(x_1 - it_1), g_2, g_3]\} + B - iC(t_1) \end{aligned} \tag{13}$$

where A, B are some constants, and $\wp$ is a Weierstrass function with invariants g_2 and g_3. Such functions are doubly periodic and have the property

$$\wp(x) = \wp(x + iK') = \wp(x + K)$$

where K' and K are the imaginary and real periods.

From (13) we have

$$p = \frac{A}{2}\{\exp[-\wp(x_1 + it_1)] + \exp[-\wp(x_1 - it_1)]\} + B \tag{14}$$

$$q = \frac{A}{2i}\{\exp[-\wp(x_1 + it_1)] - \exp[-\wp(x_1 - it_1)]\} + C(t_1)$$

Because of the double periodicity, the average of $\wp$ over a time period T_0 is identical, i.e. $\wp(x + iT_0) = \wp(x - iT_0)$, and we can write

$$p = A\exp[-\wp(x_1)] + B \tag{14a}$$

and

$$q = C(t_1)$$

Introducing (14) into (12), we get

$$\begin{aligned}\frac{4m}{\hbar^2}(V_2 - p^2 V_1) = {} & A\exp[-\wp(x_1)]\{A\exp[-\wp(x_1)] + B\}[4\wp^3 - 6\wp^2 - 4] \\ & - \frac{A^2}{2}\exp[-2\wp(x_1)]\,4[\{\wp(x_1\}^3 - 1] \\ & + \frac{2m^2}{\hbar^2}\frac{\{A\exp[-\wp(x_1)] + B - 1\}\{3 - (A\exp[-\wp(x_1)] + B)\}}{A\exp[-\wp(x_1)] + B - 2}\end{aligned} \tag{15}$$

The solution of equation (12) is further simplified by assuming the Weierstrass functions to be of the equi-anharmonic type, i.e. $g_2 = 0$. This solution is in fact identical with the mapping functions used by Ramanna (1968a, b) and gives rise, as shown later, to an array of equilateral triangles. We write the left-hand side of (15) as

$$\frac{4m}{\hbar^2}V_1(x_1)\left[\frac{V_2(x_2)}{V_1(x_1)} - p^2\right]$$

Here, $V_2(x_2)$ is the potential existing in the second domain and is a function of the coordinates of that domain. In the region of the overlap, for every value of x_2 there is a corresponding value of x_1; x_1, x_2, being coordinates of an identical point in the two systems. At this point the potentials must be identical, and we can therefore write $V_2(x_2) = SV_1(x_1)$, where S is a constant which normalises the two coordinate systems. Equation (12) now reads

$$\frac{4m}{\hbar^2} V_1(x_1) = \frac{A\{\exp(-\wp)\}p(4\wp^3 - 6\wp^2 - 4)}{S - p^2} - \frac{2A^2\{\exp(-2\wp)\}(\wp^3 - 1)}{S - p^2} + \frac{2m^2}{\hbar^2}\frac{(p-1)(p-3)}{(p-2)(p^2 - S)} \tag{16}$$

If the solution of equation (12) is an automorphic function, to which class Weierstrass functions belong, they are invariant to group transformations of the type (Davis, 1961)

$$Z = \frac{ax + b}{cx + d}$$

where a, b, c and d are constants.

We now consider two cases which are modular transformations of one another, both having real values

$$\wp_N = \wp(x, 0, 4) \tag{17a}$$

$$\wp_P = \wp(x, 0, -108) \tag{17b}$$

(17b) can also be written as

$$\wp_P = -3\wp[i\sqrt{(3)}\,x, 0, 4] \tag{18}$$

In terms of elliptic functions these two functions correspond to elliptic integrals of the first kind with modulus of sin 15° and its conjugate of sin 75°. The functions (17a) and (17b) have been tabulated by Greenhill & Haddock (1886a).

We now consider the following boundary condition concerning the two domains. At $x_1 = 0$,

$$p = \frac{\partial x_1}{\partial x_2} = A\exp(-1) + B$$

Therefore

$$\int_0^{x_1} \frac{\partial x_1}{A\exp[-\wp(x_1)] + B} = x_2 + R_0$$

where R_0 is a constant of integration which corresponds to the limit of the boundary of the second domain. At this point $\psi_2(x_2) = 0$, i.e. $\omega = 0$. In equation (7), if $\omega = 0$, $V_2 - V_1 p^2 = 0$, i.e. $[A\exp(-1) + B]^2 = V_2/V_1 = S$. In this way the values of A and B determine the value of S. In Fig. 1 the values of V_1 for different values of A and B are given. Two cases are considered, one in which the A's are positive and B's negative, and the other in which the A's are negative and B's positive. For reason to be given later, we call the former as the neutron case and the latter the proton case.

We note the following remarkable facts:

(a) It is seen that as $x \to 0$ and $x \to R_0$ for the neutron and proton case, respectively, the value of V_1 tends to $+\infty$, and this results from the first of the three terms of the right-hand side of equation (16). In this equation the first two terms involve $\hbar^2$ and can be considered to arise from quantum mechanical reasons, and we may therefore conjecture that the strong repulsive term at short distances is a quantum mechanical effect. In the second case $V_1 \to \infty$ as $x \to R_0$, a constant which results from the group transformation corresponding to a translation of the origin of coordinates compared to the first case. The general behaviour of the potential is mainly determined by the third term in the right-hand side of equation (16), as the first two terms rapidly decrease with distance.

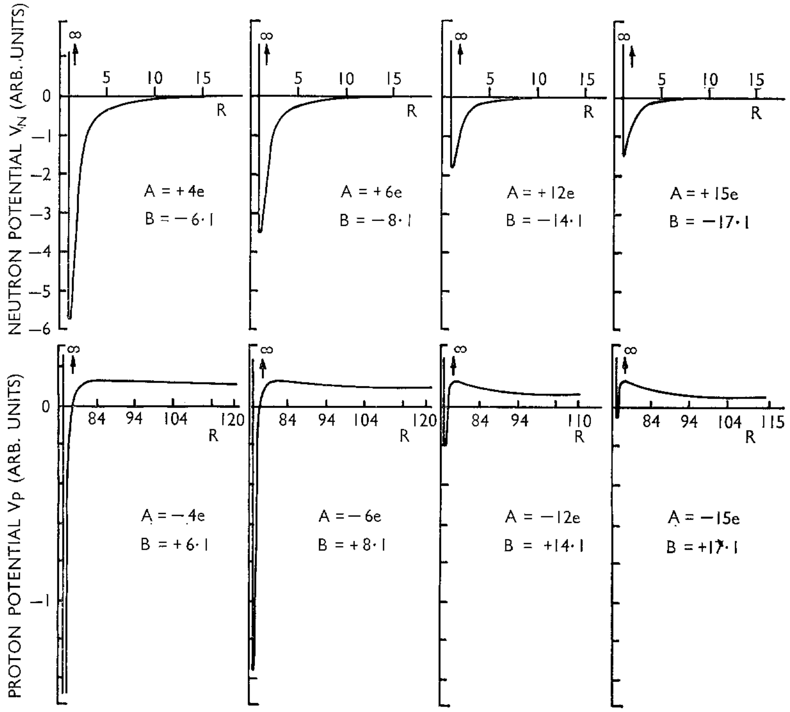

Figure 1.—The many-body neutron and proton potentials for some values of the parameters A and B.

(b) For a given value of A and B in the neutron case the potential rapidly falls from $+\infty$ becomes attractive and finally saturates to a small negative value. For the same value of A and B, but of opposite sign in the proton case, the potential rapidly falls from $+\infty$, becomes attractive, becomes repulsive again and saturates to a small positive value. For values of A

less than 16*e* the Coulomb term is clearly evident in the case of protons, while none exists for the neutron case. Hence with only one function and

TABLE 1. Values of neutron and proton potential for given values of *A* and *B*.

R	$\wp(x)$	p	V_N	R	$\wp(x)$	p	V_p
0	1·0000	−2·1000	∞	74	0·9840	—	∞
1	1·0001	−2·1025	−367·30	75	1·0548	2·7386	−0·199
2	1·0005	−2·1058	−158·26	76	1·1274	3·5345	+0·109
3	1·0012	−2·1156	−58·88	77	1·1994	4·2685	0·132
4	1·0023	−2·1253	−36·32	78	1·2720	4·9567	0·130
5	1·0033	−2.1384	−23·94	79	1·3467	5·6156	0·122
6	1·0047	−2·1547	−16·82	80	1·4232	6·2419	0·116
7	1·0065	−2·1775	−11·88	81	1·4940	6·7769	0·110
8	1·0086	−2·2036	−8·89	82	1·5693	7·3086	0·104
9	1·0110	−2·2297	−7·11	83	1·6440	7·7978	0·100
10	1·0136	−2·2623	−5·69	84	1·7202	8·2611	0·096
11	1·0165	−2·2950	−4·74	85	1·7970	8·6917	0·092
12	1·0197	−2·3341	−3·95	86	1·8747	9·0733	0·089
13	1·0232	−2·3765	−3·35	87	1·9533	9·4582	0·086
14	1·0269	−2·4189	−2·91	88	2·0331	9·6866	0·085
15	1·0310	−2·4646	−2·55	89	2·1141	10·1041	0·082
16	1·0353	−2·5168	−2·23	90	2·1960	10·4727	0·079
17	1·0400	−2·5690	−1·99	100	3·1083	12·6289	0·068
18	1·0448	−2·6244	−1·78	105	3·6546	13·2943	0·065
19	1·0501	−2·6964	−1·57	110	4·2999	13·6564	0·064
20	1·0556	−2·7484	−1·44	115	5·0583	13·9010	0·063
25	1·0877	−3·1072	−0·94	120	6·0000	14·0185	0·062
30	1·1281	−3·5443	−0·66	180		14·1000	0·062
35	1·1771	−4·0466	−0·49				
40	1·2356	−4·6175	−0·38				
50	1·3851	−5·9353	−0·34				
60	1·5875	−7·4326	−0·18				
70	1·8586	−9·0146	−0·14				
80	2·2268	−10·5813	−0·11				
100	3·4453	−13·0105	−0·08				
120	6·1072	−14·0282	−0·08				
180		−14·1000	−0·08				

Note:

$A = +12e = 32{\cdot}6196 \qquad A = -12e = -32{\cdot}6196$

$B = -14{\cdot}1 \qquad B = +14{\cdot}1$

R is given by

$$x = \left(1 - \frac{r}{180}\right)\omega_2 = \frac{R}{180}\omega_2$$

where $\omega_2 = 1{\cdot}2143$ (Greenhill, 1886a).

two constants it is possible to obtain the familiar form of the nuclear potentials. The parameters A and B are obviously connected with meson interactions and are not discussed here.

Table 1 gives the behaviour of the potential for $A = +12e$, $B = -14.1$ (neutron case) and $A = -12e$, $B = +14.1$ (proton case). It should be noted that as defined in equations (1) and (2) the V's are negative, i.e. it is attractive when V is positive and repulsive when negative. It would therefore seem that equation (16) and its implications hold only for particles of negative mass. This restriction, however, is not true, and the theory holds equally for particles of positive and negative mass. This is due to the invariance of equations (1), (2) and (12) to transformations of the type $x = -x$ and $t = -t$ if $V(-x) = -V(x)$.

Since a strong repulsion term at the origin is ascribed in nuclear theory to the Pauli principle, we conjecture that the Pauli principle arises from the geometry of the partial differential equations. To substantiate this conjecture, the spin-states will have to be introduced explicitly and it has to be shown that the effect of the principle persists even when the total spin of the system is zero. Such a possibility exists when one uses the third-type elliptical integrals as discussed by Ramanna (1968a, b).

The potentials arrived at by these geometrical descriptions give rise to a saturation effect, in that one domain can overlap with only three other domains and no more. This results from the choice of the Weierstrass functions and is discussed later. The long-range Coulomb repulsive term in the case of proton is clearly evident. The difference between the neutron and proton case arises from the fact that we have introduced a rotation of 120° equivalent to a group transformation in the respective coordinate systems, agreeing with the rotation of the mapping used by Ramanna (1968a, b) where only two such positions give rise to real solutions.

It is to be noted, that while we have considered only two Schrödinger equations, the assumption of Weierstrass functions is valid for a whole array of domains, and the nature of the array depends on the type of doubly periodic function that is assumed. In this sense the potential, given by equation (16), is a combined effect of many bodies and does not merely correspond to the two-nucleon systems. We show later that the equi-anharmonic case used above produces an array of lattice points which gives rise to a shell structure that is somewhat close to that observed in nuclear structure.

We now show that the choice of harmonic functions for the coordinate transformation automatically minimises the total energy of the system if they are described by Schrödinger equations. To show this in an elegant manner and to arrive at the energies of the systems directly without a knowledge of ψ we have to make use of the calculus of differential forms and the Dirichlet principle.

From the calculus of differential forms (Ahlfors & Sario, 1960) we have that if ω_1, is a one-form in two dimensions, i.e. $\omega_1 = a_1\,dx_1 + b_1\,dy_1$, where a_1 and b_1 are complex valued functions over some domain h_1, and if h_1

and h_2 are overlapping domains then the corresponding forms $\omega_1 = a_1 dx_1 + b_1 dy_1$ and $\omega_2 = a_2 dx_2 + b_2 dy_2$ are connected by the relations

$$a_1 = a_2 \frac{\partial x_2}{\partial x_1} + b_2 \frac{\partial y_2}{\partial x_1}$$

and (19)

$$b_1 = a_2 \frac{\partial x_2}{\partial y_1} + b_2 \frac{\partial y_2}{\partial y_1}$$

Similar relations hold for other overlapping domains. Writing in general $\omega = a dx + b dy$, the exterior derivative is given by

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx\, dy \tag{20}$$

since $dx\,dx = dy\,dy = 0$, and $dy\,dx = -dx\,dy$ by the algebra of differential forms.

If ω_1, ω_2 are defined over a Riemann surface, the following relations also hold:

$$-b_1 = -b_2 \frac{\partial x_2}{\partial x_1} + a_2 \frac{\partial y_2}{\partial x_1}$$

and

$$a_1 = -b_2 \frac{\partial x_2}{\partial y_1} + a_2 \frac{\partial y_2}{\partial y_1} \tag{19a}$$

The invariance of ω implies the invariance of the conjugate differential $\omega^* = -b dx + a dy$.

Let $\bar{\omega}^*$ be the complex conjugate of ω^*, the conjugate differential of ω, i.e. $\bar{\omega}^* = -\bar{b} dx + \bar{a} dy$, where $\bar{a}$ and $\bar{b}$ are complex conjugates of a and b respectively. Then the integral

$$I = \int_D \omega\bar{\omega}^* = \int_D (|a|^2 + |b|^2)\, dx\, dy$$

is finite and positive. It can be seen that this is a Dirichlet integral and by the Dirichlet principle it takes a minimal value if $[|a|^2 + |b|^2]$ is a harmonic function. The type of functions the integrand can take depends on the Riemann manifold chosen, i.e. one gets spherical harmonics if the Riemann surface is that of the sphere, doubly-periodic functions (elliptic integrals) if the surface is that of a torus, and other meromorphic functions for higher punctured surfaces. The Dirichlet principle also ensures that this integral is invariant under conformal transformations. If the forms ω and $\bar{\omega}^*$ are closed, then $d\omega = 0$ and $d\bar{\omega}^* = 0$.

We now write a set of Schrödinger equations and their complex conjugates into their respective differential forms and compare with the differential forms defined above. The Schrödinger equation can be written as follows

$$H_{\text{op}}\psi(x,t) - i\hbar\frac{\partial\psi(x,t)}{\partial t} = 0 \tag{21}$$

where $\psi(x,t)$ is a complex valued function of space x and time t, and H_{op} is the energy operator.

$$H_{\text{op}}\psi(x,t)\,dt\,dx - H_{\text{op}}\psi(x,t)\,dt\,dx = 0$$

Comparing this with

$$\left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)dx\,dy = 0 \tag{22}$$

(when ω is closed $d\omega = 0$), and putting $x = t$ and $y = x$, one finds

$$\frac{\partial b}{\partial t} = H_{\text{op}}\psi(x,t) = \frac{\partial a}{\partial x} \tag{23}$$

If, similarly, one compares the conjugate equation

$$[H_{\text{op}}\psi(x,t)]^*\,dt\,dx - [H_{\text{op}}\psi(x,t)]^*\,dt\,dx = 0 \tag{24}$$

with

$$\left(\frac{\partial\bar{a}}{\partial t} + \frac{\partial\bar{b}}{\partial x}\right)dt\,dx = 0$$

one gets

$$-\frac{\partial\bar{b}}{\partial x} = [H_{\text{op}}\psi(x,t)]^* = \frac{\partial\bar{a}}{\partial t} \tag{25}$$

where $[H_{\text{op}}\psi(x,t)]^*$ is the complex conjugate of $H_{\text{op}}\psi(x,t)$. It would seem that in the derivation of (23) and (25) the theory holds for all types of operators. But since a and b are just functions, the operator must be a scalar only. Hence only the energy operator is permitted.

From (23) and (25), one gets

$$\begin{aligned}
a &= \int H_{\text{op}}\psi(x,t)\,dx + f_a(t)\\
\bar{a} &= \int [H_{\text{op}}\psi(x,t)]^*\,dt + f_{\bar{a}}(x)\\
b &= \int H_{\text{op}}\psi(x,t)\,dt + f_b(x)\\
\bar{b} &= -\int [H_{\text{op}}\psi(x,t)]^*\,dx + f_{\bar{b}}(t)
\end{aligned} \tag{26}$$

Since $a\bar{a}$ and $b\bar{b}$ are real, one can get

$$f_a(t) = \int H_{\text{op}}\psi(x,t)\,dt, \qquad f_{\bar{a}}(x) = \text{complex conjugate of} \int H_{\text{op}}\psi(x,t)\,dx$$

$$f_b(x) = -\int H_{\text{op}}\psi(x,t)\,dx$$

and

$$f_{\bar{b}}(t) = \text{complex conjugate of} \int H_{\text{op}}\,\psi(x,t)\,dt$$

Rewriting

$$\begin{aligned} a &= H_x + H_t, & \bar{a} &= H_x{}^* + H_t{}^* \\ b &= -H_x + H_t, & \bar{b} &= -H_x{}^* + H_t{}^* \end{aligned} \tag{27}$$

where the notation is obvious, one gets

$$\begin{aligned} a\bar{a} &= H_x H_x{}^* + H_t H_t{}^* + H_x H_t{}^* + H_x{}^* H_t \\ b\bar{b} &= H_x H_x{}^* + H_t H_t{}^* - H_x H_t{}^* - H_x{}^* H_t \end{aligned} \tag{28}$$

so that

$$\begin{aligned} a\bar{a} + b\bar{b} &= 2(H_x H_x{}^* + H_t H_t{}^*) \\ &= 2\Big[\int H_{\text{op}}\,\psi(x,t)\,dx \int \{H_{\text{op}}\,\psi(x,t)\}^*\,dx \\ &\qquad + \int H_{\text{op}}\,\psi(x,t)\,dt \int \{(H_{\text{op}}\,\psi(x,t)\}^*\,dt\Big] \end{aligned} \tag{29}$$

The positive square root of the integral,

$$\int |a|^2 + |b|^2\,dx\,dt$$

i.e. the norm $\|\omega\|$, gives the interaction energy E within a domain. The total interaction energy is given by the sum of the norms for each domain. By the Dirichlet principle, if the integrands of I are harmonic functions the total energy is minimised. If on quantum mechanical prescriptions we had defined the a's and b's with respect to $\psi^* H_{\text{op}}\psi$ and not just $H_{\text{op}}\psi$, we would directly get

$$|a|^2 + |b|^2 = E^2$$

Since Weierstrass functions are harmonic, we can use them to minimise the Dirichlet integral. We therefore assume

$$\left.\begin{aligned} a + ib &= \wp(x + it) \\ a - ib &= \wp(x - it) \end{aligned}\right\} \tag{30}$$

where the a's and b's are defined to include the constants A and B defined in (13) and $\wp$ is the same function used earlier. If the Weierstrass functions are equi-anharmonic, one gets an array of equilateral triangles, corresponding to equal symmetrical domains in a space–time plot (Fig. 2). This is seen by writing (Greenhill, 1886b)

$$\left(\frac{1 + ir}{1 - ir}\right)^2 = -\wp[(x + it), 0, 4]$$

$$\left(\frac{1 - ir}{1 + ir}\right)^2 = -\wp[(x - it), 0, 4]$$

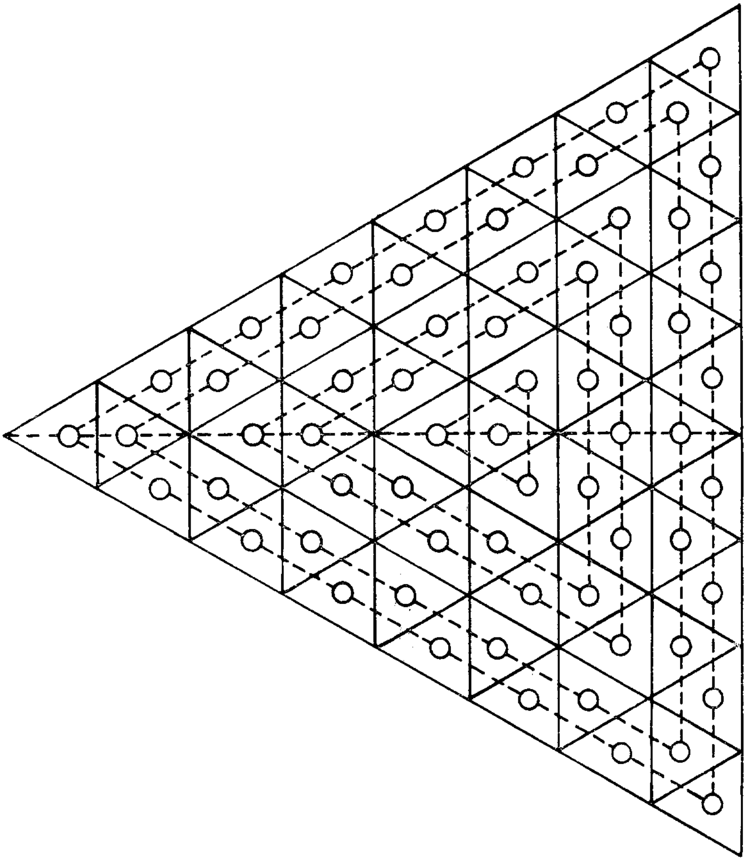

Figure 2.—Array of equilateral triangles. Centroids generated by equi-anharmonic Weierstrass functions.

and

$$x + it = \text{const} \int \frac{ds}{\sqrt{(+4s^3 - 4)}}$$

$$= d_n + d_n C_n(\alpha + i\beta) F(\theta, k) \qquad (31)$$

where d_n = distance from the centre of coordinates to one of the corners of the triangle

$(\alpha + i\beta)$ = cube roots of 1 or i, as the case may be

C_n = geometric constant determining the absolute size of the triangles

$$\left(\frac{1 - ir}{1 + ir}\right)^2 = S$$

$$\cos\theta = \frac{S - [\surd(3) + 1]}{S + [\surd(3) - 1]}$$

$k = \sin 15°$ or $\sin 75°$, depending on whether $(\alpha + i\beta)$ is the cube root root of 1 or i

Equation (31) is exactly the triangle function mapped out by Ramanna (1968a, b). Since

$$[-\wp(x + it)][-\wp(x - it)] = 1 \tag{32}$$

$(a^2 + b^2)$ is a constant and the total energy content of the domain is a constant.

From (19) and (19a) it may be deduced that the energy of the neighbouring domain is given by

$$E_2{}^2 = \int\int (a_2{}^2 + b_2{}^2)\, dx_2\, dt_2$$

$$= \int\int (a_1{}^2 + b_1{}^2)(p^2 + q^2)\, dx_2\, dt_2$$

From (30) and (32)

$$E_2{}^2 = (a_1{}^2 + b_1{}^2) \int\int \left(\frac{\partial x_1}{\partial x_2}\right)^2 + \left(\frac{\partial x_1}{\partial t_2}\right)^2 dx_2\, dt_2$$

and since

$$\int\int \left(\frac{\partial x_1}{\partial x_2}\right)^2 + \left(\frac{\partial x_1}{\partial t_2}\right)^2 dx_2\, dt_2 = \int\int dx_1\, dt_1 \tag{33}$$

$E_2{}^2 = E_1{}^2$. Equation (33) follows from the Dirichlet principle (Weyl, 1955), which also states that the integral is invariant to conformal transformations and the fact that the two coordinate systems are conformal to one another by definition.

Equation (31) produces a whole array of triangles, and the fact that each domain overlaps with three neighbours is a consequence of the three values of $\sqrt[3]{1}$ and $\sqrt[3]{i}$, as the case may be, which produce kaleidoscopic figures. Hence the choice of Weierstrass functions directly leads to a saturation effect.

In the present analysis the choice of simple Weierstrass function leads to a constant interaction energy in each domain. If, however, higher harmonic functions such as elliptic modular functions are used, one gets an array of equilateral, equiangular curvilinear triangles, decreasing asymptotically in size (Ford, 1951). This, in principle, can be used in cases where the potentials are such that they distort the geometry of the whole system.

In order to determine the Coulomb and binding energies of actual stable nuclei we consider the situation where we can associate the centroids of each of the triangles with the average position of a nucleon occupying the domain. For each average position and time x and t, respectively, there is a corresponding value of d_n and θ given by equation (31). It is seen in

Fig. 3 that the centroids can themselves be arranged in the form of concentric triangles. The centre of coordinates itself is the first centroid and is associated with two nucleons of opposite spin. In the triangle around it there are three nucleon pairs and in the third, fourth and fifth triangle shells there are, respectively, twelve, eighteen and twenty-one nucleon pairs.

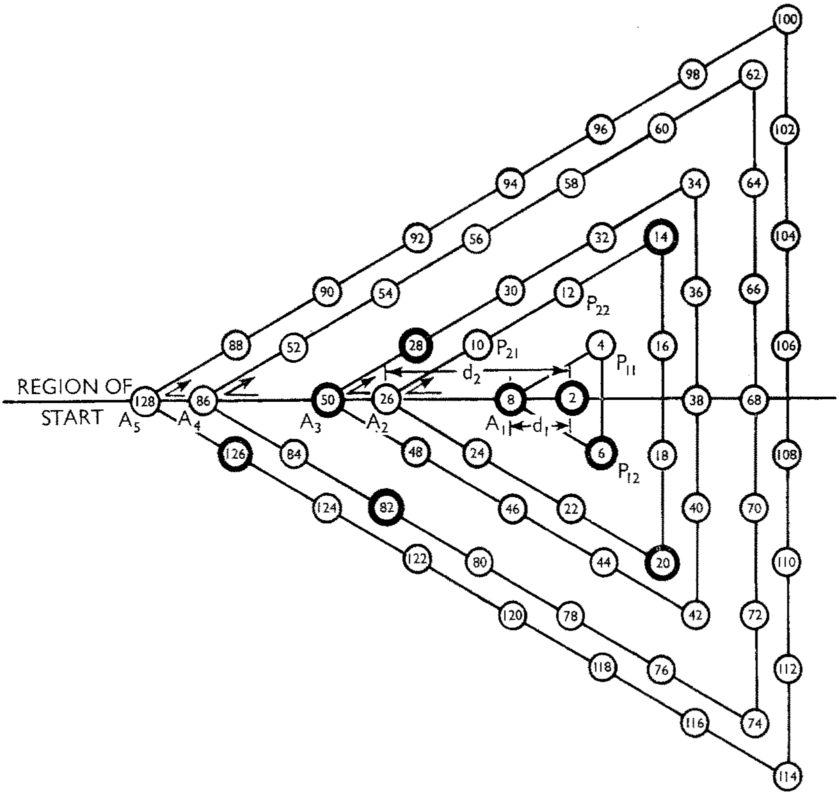

Figure 3.—Average position of nucleon pair. The distance $A_N P_N$ is a measure of the energy of the system. Magic number positions are enclosed in a dark circle.

The side of each triangle is generated as θ takes values from 0 to π and hence for each shell θ moves from 0 to π three times. We adopt a clock-wise convention to enumerate in a continuous manner the nucleon pair positions. If, therefore, we plot the number of nucleon pairs N against θ, for every range of values of θ from 0 to 3π, we get an increasing value of $(dN/d\theta)_n$ for each n, where n represents the triangle shell number. From equation (31) the amount of energy associated with each nucleon is given by

$$E = d_n C_n \sqrt{(\alpha^2 + \beta^2)} F(\theta, k) \qquad (34)$$

since for each point (x, t) occupied by a nucleon pair, the energy is increased by an equal amount. Geometrically the increase in the energy with the addition of each nucleon is equal to the distance between neighbouring centroids $A_{N1}P_{N1}$, etc. (Fig. 3).

We define another angle ϕ proportional to the nucleon pair number with a range from 0 to $\pi/2$ for the entire range of values of θ

$$\phi\alpha\left(\frac{\partial\phi}{\partial\theta}\right)\theta$$

Instead of calculating E, we derive an expression for the average energy per nucleon pair defined as follows:

$$\bar{E}=\frac{\int E\,d\phi}{\int d\phi}$$

Hence

$$\bar{E}\propto An\int dn\,F\left[\phi\Big/\frac{\partial\phi}{\partial\theta},k\right]d\phi$$

$$\propto \text{Average value of } d_n F\left[\phi\Big/\frac{\partial\phi}{\partial\theta},k\right]$$

$$\approx An\,F(\phi,k)$$

where $A_N=\sqrt{(\alpha^2+\beta^2)}\,C_n$.

Hence the average energies for a neutron and proton system taken separately are given by

$$\left.\begin{aligned}E_N&=A_N F(\phi_N,k)\\ E_P&=A_N F(\phi_P,k')\end{aligned}\right\}\qquad(35)$$

where $k=\sin 15°$ and $k'=\sin 75°$.

The systems we have considered represent an unreal situation, in as far as neutrons and protons do not form stable nuclei separately. In order to apply the theory to a more realistic situation without introducing further complications to its description, we first consider only spinless systems. By a spinless system is meant that the spin of the nucleon is neglected, and for comparison with the actual situation only indistinguishable pairs of nucleons of opposite spin are considered, and with each domain is associated a spinless nucleon pair. As shown by Ramanna (1968a, b), a solution involving a higher Weierstrass function is required if one is to consider systems with spin. We also make the following assumptions.

(a) Nuclear forces are charge independent, i.e. the total interaction energies of the neutron and proton systems are identical except for the Coulomb charge.

(b) By considering only time independent situations all information about radioactive nuclei is lost. We have therefore to assume that the range of values of ϕ which lie between 0 and $\pi/2$ is restricted to known stable nuclei, i.e. the maximum value of the neutron $N_0 = 126$ and the maximum value of the proton number $Z_0 = 82$. We therefore write

$$\phi_{\text{neutron}} = \frac{N-2}{N_0-2}\pi/2 \tag{36}$$

and

$$\phi_{\text{proton}} = \frac{Z-2}{Z_0-2}K\pi/2 \tag{37}$$

where K is some constant. $(N-2)$ and $(Z-2)$ are used instead of N or Z to account for the fact that the first stable spinless system is the He^4 nucleus.

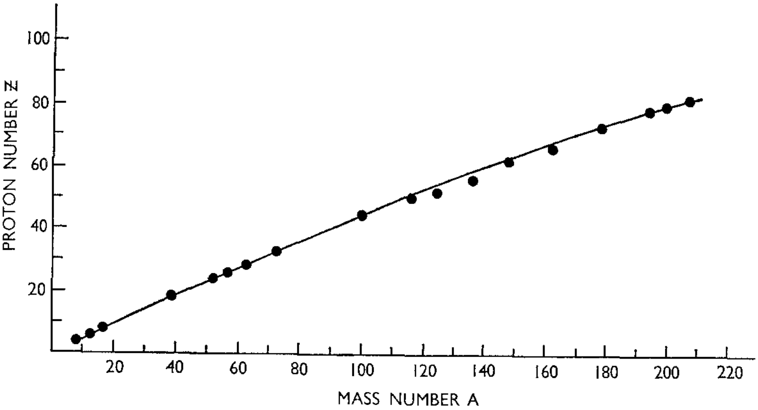

Figure 4.—The neutron–proton ratios of β-stable nuclei. The continuous line shows the theoretical predictions.

(c) By the use of the Dirichlet principle, the energies of the neutron and proton system have been minimised separately and it is now necessary to consider the condition of stability when the two systems are coupled to form stable nuclei. This stability must correspond to the coupled system having the minimum potential energy. In the coupled system, E_N is a function of Z and E_P is a function of N. For a given neutron number $\partial/\partial z(E_N + E_P) = 0$, i.e. $E_N + E_P$ a constant. In order to obtain all the β-stable nuclei, we set $E_N + E_P = 0$. Figure 4 gives the neutron–proton ratios for the value of $K = 0{\cdot}76$ from the relation

$$F(\phi_N, 15^\circ) = F(\phi_P, 75^\circ) \tag{38}$$

with the ϕ's defined as in (37). Since the triangular domains for proton are rotated by 60° with respect to the neutron triangles, each coupled neutron–proton pair gives rise to a star.

It is clear that if $E_N + E_P$ had been chosen to be a constant other than zero, depending on the value of this constant, it is possible to obtain families of nuclei which are neighbours to the β-stable nuclei.

We interpret equation (35) as to give the energy of a proton with a nuclear term E_P' and a Coulomb term E_C, i.e.

$$E_C + E_P' = A_N F(\phi_p, 75^\circ)$$

and

$$E_C - E_C(\mathrm{He}^4) = A_N F(\phi_p, 75^\circ) - E_P'$$

It is necessary to subtract the Coulomb energy of the He^4 nucleus from the total Coulomb energy, since the interaction energy of the He^4 nucleus is undefined, being the first of the spinless systems. From the assumption of charge-independence we write

$$E_P' = A_P F(\phi_p, 15^\circ)$$

which means that the behaviour of the proton energy when devoid of its charge is identical with that of the neutron energy—except for the normalisation constant, which we write as A_P instead of A_N. Therefore

$$\begin{aligned} E_C - E_C(\mathrm{He}^4) &= Q[A_N F(\phi_N, 15^\circ) - A_P F(\phi_P, 15^\circ)] \\ &\simeq Q[A_N \phi_N - A_P \phi_P] \end{aligned} \tag{39}$$

since $F(\phi, 15^\circ) \approx \phi$; and Q is a constant of normalisation.

We define the binding energy as that part of the nuclear energy from each system which contributes to keeping the system together. The rest of the nuclear energy is used to compensate the Coulomb repulsion. Hence we write

$$E_B - E_B(\mathrm{He}^4) = \mu_1 E_N + \mu_2 E_P' \tag{40}$$

where E_B = the binding energy

$E_B(\mathrm{He}^4)$ = the binding energy of the He^4 nucleus which is subtracted for reasons given earlier

μ_1 and μ_2 = two fractions which determine the amount of nuclear energy available for binding from each system respectively.

We assume that

$$K \frac{A_P}{A_N} \frac{(N_0 - 2)}{(Z_0 - 2)} = 1 \tag{41}$$

i.e. the ratio A_N/A_P of the geometric constants A_N and A_P, which determine the absolute size of the domains gives the maximum neutron and proton numbers directly.

Equation (39) becomes

$$E_C - E_C(\mathrm{He}^4) = \frac{QA_N}{N_0 - 2} \frac{\pi}{2} (N - Z) \tag{42}$$

TABLE 2. Experimental and theoretical values of Coulomb and binding energies

Experimental				Theoretical						
A	N	Z	E_B	Z_{cal}	A_{cal}	$(N-Z)_{\text{cal}}$	E_C	$\frac{E_C}{\left[Z(Z-1)\big/A^{1/3}-\frac{2}{\sqrt[3]{4}}\right]}$	Normalised values of r_0 (approximate)	E_B
8	4	4	—	3·6974	7·6974	0·3026	0·3561	0·0751	—	—
12	6	6	92·136	5·3943	11·3943	0·6057	0·7147	0·0622	—	93·786
16	8	8	127·584	7·0874	15·0874	0·9126	1·0670	0·0509	0·75	126·462
38	20	18	327·256	17·1660	37·1660	2·8340	3·0128	0·0336	1·09	321·067
48	26	22	418·656	22·1042	48·1042	3·8958	3·8743	0·0308	1·19	416·538
52	28	24	456·248	23·7316	51·7316	4·2684	4·3322	0·0295	1·24	448·023
56	30	26	492·184	25·3487	55·3487	4·6513	4·8161	0·0286	1·28	479·322
62	34	28	545·538	28·5400	62·5400	5·4600	5·3251	0·0281	1·30	541·142
72	40	32	628·992	33·2288	73·2288	6·7712	5·8682	0·0247	1·47	600·704
100	56	44	—	44·9062	100·9062	11·0938	9·8386	0·0242	1·50	—
116	66	50	987·972	51·8284	117·8284	14·1716	12·5039	0·0250	1·45	995·084
124	72	52	1050·528	55·6423	127·6423	16·3577	14·2691	0·0269	1·35	1070·185
136	80	56	1134·512	60·4805	140·4805	19·5195	16·5929	0·0278	1·31	1165·844
148	86	62	1225·588	63·9000	149·9000	22·1000	21·4209	0·0300	1·21	1233·798
162	96	66	1324·836	69·2104	165·2104	26·7896	23·9546	0·0305	1·19	1340·013
178	106	72	1432·544	74·0570	180·0570	31·9430	29·7541	0·0328	1·11	1437·824
194	116	78	1539·390	78·4660	194·4660	37·5340	36·9462	0·0357	1·02	1527·707
200	120	80	1580·600	80·1160	200·1160	39·8840	39·7137	0·0368	0·99	1561·601
208	126	82	1636·900	82·4749	208·4749	43·5251	42·7969	0·0382	0·95	1610·340

Note: (i) The values of Z have been calculated by using equation (38).

(ii) E_C is obtained by interpolation from the graph $(N-Z)_{\text{cal}}$ versus A_{cal}.

(iii) r_0 has been obtained by neglecting $E_C(\text{He}^4)$ and normalising it to 1·5 at $A = 100$.

The unnormalised values of E_C are given in Table 2 and compared with the classical values of $Z(Z-1)/r_0A^{1/3}$ predicting approximately the usual, empirically assumed variation in the value of radius parameter r_0. In column 10 of Table 2, the values of r_0 are given when $E_C(\mathrm{He}^4)$ is neglected.

Using equations (36), (37) and (41), equation (40) becomes

$$E_B - E_B(\mathrm{He}^4) = \mu_1 A_N \phi_N + \mu_2 A_P \phi_P$$
$$= \frac{\mu_1 A_N}{N_0 - 2}\frac{\pi}{2}\left[(N-2) + \frac{\mu_2}{\mu_1}(Z-2)\right] \tag{43}$$

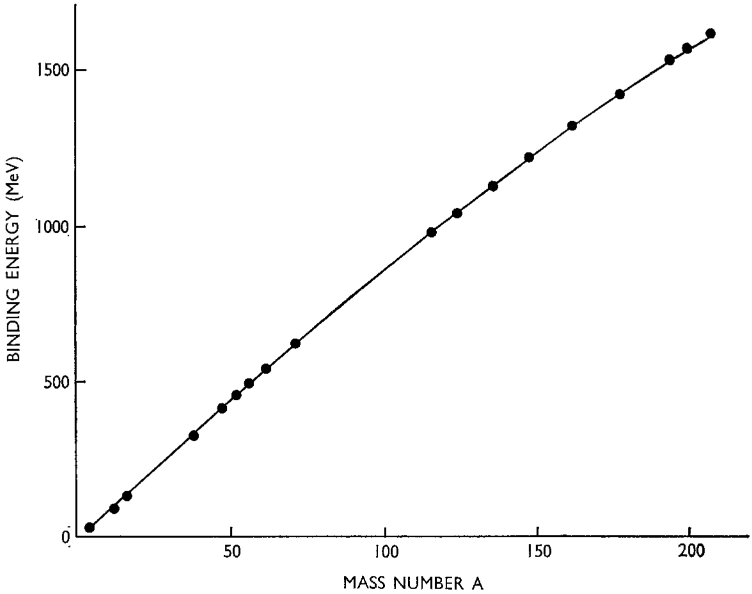

Figure 5.—The binding energies predicted by the theory compared with experimental points.

The values of E_B are given in Table 2 and in Fig. 5, with the assumption

$$\frac{\mu_1 A_N}{N_0 - 2}\frac{\pi}{2} = 1 \quad \text{and} \quad \frac{\mu_1}{\mu_2} = \frac{1}{18}$$

The Coulomb energy can also be written as

$$E_C - E_C(\mathrm{He}^4) = (1-\mu_1)E_N + \alpha(1-\mu_2)E_P' \tag{44}$$

which asserts that of the total nuclear energy available, part of it is used to compensate the repulsive Coulomb energy. The constant α is introduced to take into account that this compensation comes differently from the two systems. Equation (44) also defines the binding energy in the usual form $E_B = aE_N + bE_P' - cE_C$.

Comparing equation (39) with equation (44), and using equation (41), we get

$$Q(A_N \phi_N - A_P \phi_P) = (1 - \mu_1) A_N \phi_N + \alpha(1 - \mu_2) A_P \phi_P$$

i.e.

$$\frac{A_N \phi_N}{A_P \phi_P} = \frac{Q + \alpha(1 - \mu_2)}{Q - 1 + \mu_1}$$

As Q, α, μ_1 and μ_2 are constants independent of ϕ_N and ϕ_P,

$$Q = 1 - \mu_1 = \alpha(\mu_2 - 1) \tag{45}$$

since

$$\frac{\mu_1 A_N}{N_0 - 2} \frac{\pi}{2} = 1$$

if A_N is known, μ_1 and Q are determined. If A_P is also known, K is determined from equation (41). Hence if A_N, A_P and α are known, all the constants are determined. Since the time dependence of the equations has been neglected, the maximum neutron and proton number $N_0 = 126$ and $Z_0 = 82$ of stable nuclei have been assumed.

The fits are also consistent with the value of $E_B(\mathrm{He}^4) = 28{\cdot}288$ MeV. No attempt has been made to get exact fits to experimental results, as this would not be justified in view of the very simple assumptions involved. The comparisons are made to show the closeness of the predictions to the measured values in spite of the very general nature of the theory.

It is seen from the enumeration of the lattice points (Fig. 3) that a shell structure is evident. For nucleon numbers greater than 28 there is only one magic number in each shell. While they do not all fall on the corners of the triangles, they almost all seem to do so. This structure would have shown itself in the binding energies if we had chosen E and not $\bar{E}$ from equation (35). These results have been obtained from a chosen geometrical structure of the Schrödinger equations, which is consistent with the usual nuclear potential. It is possible that a higher geometry will predict the position of the shell numbers even more exactly.

The extension of the theory to systems with spin, radioactive (time dependent) nuclei and atomic structure is being investigated. The main purpose of this work is to show that nuclear problems can be studied without reference to any models but with the choice of appropriate manifolds which, as shown here, are strictly equivalent to many body potentials. The present method, which is in the spirit of the Erlanger programme of Felix Klein (1872), seems to offer a powerful and elegant approach to many-body problems in general, and in particular a simple and elegant method of studying nuclear many-body problems, where the very existence of short-range forces probably implies a simple geometry.

Appendix

In equation (6) if we assume that $\partial\omega/\partial t_1 = 0$ we get

$$\frac{1}{\omega}\frac{\partial\omega}{\partial x_1} + \frac{1}{2p}\frac{\partial p}{\partial x_1} - \frac{1}{2q}\frac{\partial p}{\partial t_1} = \frac{mi}{\hbar q}(1-p) \tag{A.1}$$

$$\frac{q}{p}\frac{1}{\omega}\frac{\partial\omega}{\partial x_1} + \frac{q}{2p^2}\frac{\partial p}{\partial x_1} - \frac{1}{2p}\frac{\partial p}{\partial t_1} = \frac{mi}{\hbar p}(1-p) \tag{A.2}$$

Differentiating (A.1) with respect to x_1, (A.2) with respect to t_1 and adding, we get

$$\begin{aligned}\frac{1}{\omega}\frac{\partial^2\omega}{\partial x_1{}^2} + \frac{2q}{p}\frac{1}{\omega}\frac{\partial^2\omega}{\partial x_1\,\partial t_1} &= \frac{1}{\omega^2}\left(\frac{\partial\omega}{\partial x_1}\right)^2 - \frac{2}{\omega}\frac{\partial\omega}{\partial x_1}\left(\frac{1}{p}\frac{\partial p}{\partial x_1} - \frac{q}{p^2}\frac{\partial p}{\partial t_1}\right) - \frac{1}{2p}\frac{\partial^2 p}{\partial x_1{}^2}\\ &\quad - \frac{1}{2p^2}\left(\frac{\partial p}{\partial x_1}\right)^2 - \frac{mi}{\hbar q}\frac{\partial p}{\partial x_1} + \left(-\frac{1}{p^2} + \frac{1}{2q^2}\right)\left(\frac{\partial p}{\partial t_1}\right)^2 + \frac{1}{p}\frac{\partial^2 p}{\partial t_1{}^2}\\ &\quad + \left(\frac{1}{2q} - \frac{q}{p^2}\right)\frac{\partial^2 p}{\partial x_1\,\partial t_1} + \frac{2q}{p^3}\frac{\partial p}{\partial x_1}\frac{\partial p}{\partial t_1} + \frac{mi}{\hbar q^2}(1-p)\frac{\partial p}{\partial t_1} - \frac{2mi}{\hbar p^2}\frac{\partial p}{\partial t_1}\end{aligned}$$

Substituting in (7) we get

$$\begin{aligned}&\frac{1}{\omega^2}\left(\frac{\partial\omega}{\partial x_1}\right)^2 - \frac{2}{\omega}\frac{\partial\omega}{\partial x_1}\left\{\frac{1}{p}\frac{\partial p}{\partial x_1} - \frac{q}{p^2}\frac{\partial p}{\partial t_1}\right\} - \frac{1}{2p}\frac{\partial^2 p}{\partial x_1{}^2} - \frac{1}{2p^2}\left(\frac{\partial p}{\partial x_1}\right)^2\\ &- \frac{mi}{\hbar q}\frac{\partial p}{\partial x_1} + \left(-\frac{1}{p^2} + \frac{1}{2q^2}\right)\left(\frac{\partial p}{\partial t_1}\right)^2 + \frac{1}{p}\frac{\partial^2 p}{\partial t_1{}^2} + \left(\frac{1}{2q} - \frac{q}{p^2}\right)\frac{\partial^2 p}{\partial x_1\,\partial t_1}\\ &+ \frac{2q}{p^3}\frac{\partial p}{\partial x_1}\frac{\partial p}{\partial t_1} + \frac{mi}{\hbar q^2}(1-p)\frac{\partial p}{\partial t_1} - \frac{2mi}{\hbar p^2}\frac{\partial p}{\partial t_1} + \frac{2m}{\hbar^2 p^2}(V_2 - p^2 V_1)\\ &\qquad + \frac{1}{\omega}\frac{\partial\omega}{\partial x_1}\left(\frac{1}{p}\frac{\partial p}{\partial x_1} + \frac{q}{p^2}\frac{\partial p}{\partial t_1} + \frac{2mqi}{\hbar p^2}\right) = 0\end{aligned} \tag{A.3}$$

Eliminating $\partial\omega/\partial x_1$ using (A.1), we get, after some simplification,

$$\begin{aligned}&\frac{1}{4p^2}\left(\frac{\partial p}{\partial x_1}\right)^2 - \frac{1}{2p}\frac{\partial^2 p}{\partial x_1{}^2} + \frac{m^2}{\hbar^2 p^2 q^2}(1-p)(p^3 - p^2 - 2q^2) - \frac{2p^2 - q^2}{2p^3 q}\frac{\partial p}{\partial x_1}\frac{\partial p}{\partial t_1}\\ &+ \frac{2m}{\hbar^2 p^2}(V_2 - p^2 V_1) + \frac{mi}{\hbar p^3 q}\frac{\partial p}{\partial x_1}(p^3 - 2p^2 - q^2) + \frac{p^2 - 2q^2}{2qp^2}\frac{\partial^2 p}{\partial x_1\,\partial t_1}\\ &+ \frac{(3p^2 + 2q^2)}{4p^2 q^2}\left(\frac{\partial p}{\partial t_1}\right)^2 + \frac{1}{p}\frac{\partial^2 p}{\partial t_1{}^2} + \frac{mi}{\hbar p^2 q^2}\{2(p^2 + q^2) - p(2p^2 + 3q^2)\}\frac{\partial p}{\partial t_1} = 0\end{aligned} \tag{A.4}$$

In a time-independent case let us assume that

$$\frac{1}{p}\frac{\partial^2 p}{\partial {t_1}^2}+\frac{(3p^2+2q^2)}{2p^2q^2}\left(\frac{\partial p}{\partial t_1}\right)^2+\frac{p^2-2q^2}{2qp^2}\frac{\partial^2 p}{\partial x_1\,\partial t_1}$$

$$-\frac{2p^2-q^2}{2p^3q}\frac{\partial p}{\partial x_1}\frac{\partial p}{\partial t_1}+\frac{mi}{\hbar p^2q^2}\{2(p^2+q^2)-p(2p^2+3q^2)\}\frac{\partial p}{\partial t_1}=0$$

so that (A.4) becomes

$$p\frac{\partial^2 p}{\partial {x_1}^2}-\frac{1}{2}\left(\frac{\partial p}{\partial x_1}\right)^2+\frac{2m^2}{\hbar^2q^2}(p-1)(p^3-p^2-2q^2)$$

$$-\frac{2mi}{\hbar pq}\frac{\partial p}{\partial x_1}(p^3-2p^2-q^2)=\frac{4m}{\hbar^2}(V_2-p^2V_1) \quad \text{(A.5)}$$

Equation (A.5) split up into real and imaginary parts yields equations (10) and (11).

References

Ahlfors, L. V. and Sario, S. (1960). *Riemann Surfaces*. Princeton University Press.

Davis, T. P. (1961). *Introduction to Nonlinear Differential and Integral Equations*. United States Atomic Energy Commission.

Flanders, H. (1963). *Differential Forms*. Academic Press, New York.

Ford, L. E. (1951). *Automorphic Functions*. Chelsea Publishing Company, New York.

Greenhill, A. G. (1886a). *Proceedings of the London Mathematical Society*, **17**, 267.

Greenhill, A. G. (1886b). *Proceedings of the London Mathematical Society*, **17**, 363.

Hermann, R. (1965). *Advances in Mathematics*. Academic Press, New York.

Ramanna, R. (1968a). *Proceedings of the Indian Academy of Sciences*, Vol. LXVII, 328.

Ramanna, R. (1968b). *BARC Report* 350.

Weyl, H. (1955). *The Concept of a Riemann Surface*. Addison-Wesley, Reading, Mass.